

Some Lagrangians with Zeta Function Nonlocality

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Abstract

Some nonlocal and nonpolynomial scalar field models originated from p -adic string theory are considered. Infinite number of space-time derivatives is governed by the Riemann zeta function through d'Alembertian \square in its argument. Construction of the corresponding Lagrangians begins with the exact Lagrangian for effective field of p -adic tachyon string, which is generalized replacing p by arbitrary natural number n and then taken a sum of over all n . Some basic classical field properties of these scalar fields are obtained. In particular, some trivial solutions of the equations of motion and their tachyon spectra are presented. Field theory with Riemann zeta function nonlocality is also interesting in its own right.

*Dedicated to Professor I.L. Buchbinder
on the occasion of his 60th anniversary*

1 Introduction

The first paper on a p -adic string is published in 1987 [1]. After that various p -adic structures have been observed not only in string theory but also in

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many other models of modern mathematical physics (for a review of the early days developments, see e.g. [2, 3]).

One of the remarkable achievements in p -adic string theory is construction of a field model for open scalar p -adic string [4, 5]. The effective tachyon Lagrangian is very simple and exact. It describes four-point scattering amplitudes as well as all higher ones at the tree-level.

This field theory approach to p -adic string theory has been significantly pushed forward when was shown [6] that it may describes tachyon condensation and brane descent relations. After this success, many aspects of p -adic string dynamics have been investigated and compared with dynamics of ordinary strings (see, e.g. [7, 8, 9, 10] and references therein). Noncommutative deformation of p -adic string world-sheet with a constant B-field was investigated in [11] (on p -adic noncommutativity see also [12]). A systematic mathematical study of spatially homogeneous solutions of the relevant nonlinear differential equations of motion has been of considerable interest (see [9, 13, 14, 15] and references therein). Some possible cosmological implications of p -adic string theory have been also investigated [16, 17, 18, 19, 20]. It was proposed [21] that p -adic string theories provide lattice discretization to the world-sheet of ordinary strings. As a result of these developments, some nontrivial features of ordinary string theory have been reproduced from the p -adic effective action. Moreover, there have been established many similarities and analogies between p -adic and ordinary strings.

Adelic approach to the string scattering amplitudes enables to connect p -adic and ordinary counterparts ([2, 3] as a review, and see also [22]). Moreover, it eliminates unwanted prime number parameter p contained in p -adic amplitudes and also cures the problem of p -adic causality violation. Adelic generalization of quantum mechanics was also successfully formulated, and it was found a connection between adelic vacuum state of the harmonic oscillator and the Riemann zeta function [23]. Recently, an interesting approach toward foundation of a field theory and cosmology based on the Riemann zeta function was proposed in [24]. Note that p -adic and ordinary sectors of the four point adelic string amplitudes separately contain the Riemann zeta function (see, e.g. [2], [3] and [25]).

The present paper is mainly motivated by our intention to obtain the corresponding effective Lagrangian for adelic scalar string. Hence, as a first step we investigate possibilities to derive Lagrangian related to the p -adic sector of adelic string. Starting with the exact Lagrangian for the effective field of p -adic tachyon string, extending prime number p to arbitrary natural

number n and undertaking various summations of such Lagrangians over all n , we obtain some scalar field models with the operator valued Riemann zeta function. Emergence of the Riemann zeta function at the classical level can be regarded as its analog of quantum scattering amplitude. This zeta function controls spacetime nonlocality. In the sequel we shall construct and explore some classical field models which should help in investigation of some properties of adelic scalar strings.

2 Construction of zeta nonlocal Lagrangians

The exact tree-level Lagrangian of effective scalar field φ for open p -adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \varphi p^{-\frac{\square}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right], \quad (1)$$

where p is any prime number, $\square = -\partial_t^2 + \nabla^2$ is the D -dimensional d'Alembertian.

The equation of motion for (1) is

$$p^{-\frac{\square}{2m_p^2}} \varphi = \varphi^p, \quad (2)$$

and its properties have been studied by many authors (see e.g. [9, 13, 14, 15] and references therein).

Prime number p in (1) and (2) can be replaced by any natural number $n \geq 2$ and such expressions also make sense. Moreover, if $p = 1 + \varepsilon \rightarrow 1$ there is the limit of (1)

$$\mathcal{L} = \frac{m^D}{g^2} \left[\frac{1}{2} \varphi \frac{\square}{m^2} \varphi + \frac{\varphi^2}{2} (\ln \varphi^2 - 1) \right] \quad (3)$$

which corresponds to the ordinary bosonic string in the boundary string field theory [26].

Now we want to introduce a model which incorporates all the above string Lagrangians (1) with p replaced by $n \in \mathbb{N}$. To this end, we take the sum of all Lagrangians \mathcal{L}_n in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[-\frac{1}{2} \phi n^{-\frac{\square}{2m_n^2}} \phi + \frac{1}{n+1} \phi^{n+1} \right], \quad (4)$$

whose explicit realization depends on particular choice of coefficients C_n , string masses m_n and coupling constants g_n . To avoid a divergence problem in $1/(n-1)$ when $n = 1$ one has to take that $C_n m_n^D/g_n^2$ is proportional to $n-1$. In this paper we shall consider a case when coefficients C_n are proportional to $n - 1$, while masses m_n as well as coupling constants g_n do not depend on n , i.e. $m_n = m$, $g_n = g$. Since this is an approach towards effective Lagrangian of an adelic string it seems natural to take mass and coupling constant independent on particular p or n . To emphasize that Lagrangian (4) describes a new field, which is different from a particular p -adic one, we introduced notation ϕ instead of φ . The two terms in (4) with $n = 1$ are equal up to the sign, but we remain them because they provide the suitable form of total Lagrangian L .

2.1 Case $C_n = \frac{n-1}{n^{2+h}}$

Let us first consider the case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (5)$$

where h is a real number. The corresponding Lagrangian is

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} n^{-\frac{\square}{2m^2}-h} \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right] \quad (6)$$

and it depends on parameter h .

According to the famous Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\square}{2m^2}-h} = \prod_p \frac{1}{1 - p^{-\frac{\square}{2m^2}-h}}.$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (7)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (7) we can rewrite (6) in the form

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta \left(\frac{\square}{2m^2} + h \right) \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right]. \quad (8)$$

Here $\zeta \left(\frac{\square}{2m^2} + h \right)$ acts as a pseudodifferential operator

$$\zeta \left(\frac{\square}{2m^2} + h \right) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta \left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk, \quad (9)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)} \phi(x) dx$ is the Fourier transform of $\phi(x)$. Lagrangian L_0 , with the restriction on momenta $-k^2 = k_0^2 - \vec{k}^2 > (2 - 2h)m^2$ and field $|\phi| < 1$, is analyzed in [27]. In the sequel we shall consider Lagrangian (8) with analytic continuations of the zeta function and the power series $\sum \frac{n^{-h}}{n+1} \phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta \left(\frac{\square}{2m^2} + h \right) \phi + \mathcal{AC} \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right], \quad (10)$$

where \mathcal{AC} denotes analytic continuation.

Nonlocal dynamics of this field ϕ is encoded in the pseudodifferential form of the Riemann zeta function. When the d'Alembertian is in the argument of the Riemann zeta function we say that we have zeta nonlocality. Accordingly, this ϕ is a zeta nonlocal scalar field.

Potential of the above zeta scalar field (10) is equal to $-L_h$ at $\square = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2} \left(\frac{\phi^2}{2} \zeta(h) - \mathcal{AC} \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right), \quad (11)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ -function vanishes at $h = -2, -4, -6, \dots$.

The equation of motion in differential and integral form is

$$\zeta \left(\frac{\square}{2m^2} + h \right) \phi = \mathcal{AC} \sum_{n=1}^{+\infty} n^{-h} \phi^n, \quad (12)$$

$$\frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} e^{ixk} \zeta \left(-\frac{k^2}{2m^2} + h \right) \tilde{\phi}(k) dk = \mathcal{AC} \sum_{n=1}^{+\infty} n^{-h} \phi^n, \quad (13)$$

respectively. It is clear that $\phi = 0$ is a trivial solution for any real h . Existence of other trivial solutions depends on parameter h . When $h > 1$ we have another constant trivial solution $\phi = 1$.

In the weak field approximation ($|\phi(x)| \ll 1$) the above expression (13) becomes

$$\int_{\mathbb{R}^D} e^{ikx} \left[\zeta \left(-\frac{k^2}{2m^2} + h \right) - 1 \right] \tilde{\phi}(k) dk = 0, \quad (14)$$

which has a solution $\tilde{\phi}(k) \neq 0$ if equation

$$\zeta \left(\frac{-k^2}{2m^2} + h \right) = 1 \quad (15)$$

is satisfied. According to the usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$, equation (15) in the form

$$\zeta \left(\frac{M^2}{2m^2} + h \right) = 1, \quad (16)$$

determines mass spectrum $M^2 = \mu_h m^2$, where set of values of spectral function μ_h depends on h .

Equation (16) gives infinitely many tachyon mass solutions. Namely, function $\zeta(s)$ is continuous for real $s \neq 1$ and changes sign crossing its zeros $s = -2n$, $n \in \mathbb{N}$. According to relation $\zeta(1-2n) = -B_{2n}/(2n)$ and values of the Bernoulli numbers ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, $B_{14} = 7/6$, $B_{16} = -3617/510$, $B_{18} = 43867/798$, \dots) it follows that $|\zeta(1-2n)| = |B_{2n}/(2n)| > 1$ if and only if $n \geq 9$. Taking into account also regions where $\zeta(1-2n) > 0$ we conclude that $\zeta(s) = 1$ has two solutions when $-20 - 4j < s < -18 - 4j$ for every $j = 0, 1, 2, \dots$. Consequently, for any $h \in \mathbb{Z}$, we obtain infinitely many tachyon masses M^2 :

$$M^2 = -(40 + 8j + 2h - a_j) m^2 \quad \text{and} \quad M^2 = -(36 + 8j + 2h + b_j) m^2, \quad (17)$$

where $a_j \ll 1$, $b_j \ll 1$ and $j = 0, 1, 2, \dots$.

An elaboration of the above Lagrangian for $h = 0, \pm 1, \pm 2$ is presented in [28].

2.2 Case $C_n = \frac{n^2-1}{n^2}$

In this case Lagrangian (4) becomes

$$L = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} \left(n^{-\frac{\square}{2m^2}+1} + n^{-\frac{\square}{2m^2}} \right) \phi + \sum_{n=1}^{+\infty} \phi^{n+1} \right] \quad (18)$$

and it yields

$$L = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \left\{ \zeta\left(\frac{\square}{2m^2} - 1\right) + \zeta\left(\frac{\square}{2m^2}\right) \right\} \phi + \frac{\phi^2}{1-\phi} \right]. \quad (19)$$

The corresponding potential is

$$V(\phi) = -\frac{m^D}{g^2} \frac{31-7\phi}{24(1-\phi)} \phi^2, \quad (20)$$

which has the following properties: $V(0) = V(31/7) = 0$, $V(1 \pm 0) = \pm\infty$, $V(\pm\infty) = -\infty$. At $\phi = 0$ potential has local maximum.

The equation of motion is

$$\left[\zeta\left(\frac{\square}{2m^2} - 1\right) + \zeta\left(\frac{\square}{2m^2}\right) \right] \phi = \frac{\phi((\phi-1)^2+1)}{(\phi-1)^2}, \quad (21)$$

which has only $\phi = 0$ as a constant real solution. Its weak field approximation is

$$\left[\zeta\left(\frac{\square}{2m^2} - 1\right) + \zeta\left(\frac{\square}{2m^2}\right) - 2 \right] \phi = 0, \quad (22)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2} - 1\right) + \zeta\left(\frac{M^2}{2m^2}\right) = 2. \quad (23)$$

From (23) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79 m^2$ and many tachyon solutions when $M^2 < -38 m^2$.

3 Extension by ordinary Lagrangian

Let us now add ordinary bosonic Lagrangian (3) to the above constructed ones, i.e. $\mathbf{L}_h = L_h + \mathcal{L}$ and $\mathbf{L} = L + \mathcal{L}$.

Respectively, one has Lagrangian, potential, equation of motion and mass spectrum condition:

$$\mathbf{L}_h = \frac{m^D}{g^2} \left[\frac{\phi}{2} \left\{ \frac{\square}{m^2} - \zeta \left(\frac{\square}{2m^2} + h \right) \right\} \phi + \frac{\phi^2}{2} (\ln \phi^2 - 1) + \mathcal{AC} \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right], \quad (24)$$

$$\mathbf{V}_h(\phi) = \frac{m^D}{g^2} \left[\frac{\phi^2}{2} \left(\zeta(h) + 1 - \ln \phi^2 \right) - \mathcal{AC} \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right], \quad (25)$$

$$\left[\zeta \left(\frac{\square}{2m^2} + h \right) - \frac{\square}{m^2} \right] \phi = \phi \ln \phi^2 + \mathcal{AC} \sum_{n=1}^{+\infty} \frac{\phi^n}{n^h}, \quad (26)$$

$$\zeta \left(\frac{M^2}{2m^2} + h \right) - \frac{M^2}{m^2} = -1. \quad (27)$$

An analysis of these expressions depending on parameter h will be presented elsewhere. When $C_n = \frac{n^2-1}{n^2}$ one respectively obtains:

$$\mathbf{L} = \frac{m^D}{g^2} \left[\frac{\phi}{2} \left\{ \frac{\square}{m^2} - \zeta \left(\frac{\square}{2m^2} - 1 \right) - \zeta \left(\frac{\square}{2m^2} \right) - 1 \right\} \phi + \frac{\phi^2}{2} \ln \phi^2 + \frac{\phi^2}{1-\phi} \right], \quad (28)$$

$$\mathbf{V}(\phi) = \frac{m^D}{g^2} \frac{\phi^2}{2} \left[\zeta(-1) + \zeta(0) + 1 - \ln \phi^2 - \frac{1}{1-\phi} \right], \quad (29)$$

$$\left[\zeta \left(\frac{\square}{2m^2} - 1 \right) + \zeta \left(\frac{\square}{2m^2} \right) - \frac{\square}{m^2} + 1 \right] \phi = \phi \ln \phi^2 + \phi + \frac{2\phi - \phi^2}{(1-\phi)^2}, \quad (30)$$

$$\zeta \left(\frac{M^2}{2m^2} - 1 \right) + \zeta \left(\frac{M^2}{2m^2} \right) = \frac{M^2}{m^2}. \quad (31)$$

Potential (29) has one local minimum $\mathbf{V}(0) = 0$ and two local maxima, which are approximately: $\mathbf{V}(-0.6) \approx 0.15 \frac{m^D}{g^2}$ and $\mathbf{V}(0.3) \approx 0.06 \frac{m^D}{g^2}$. It has also the following properties: $\mathbf{V}(1 \pm 0) = \pm \infty$ and $\mathbf{V}(\pm \infty) = -\infty$.

In addition to many tachyon solutions, equation (31) has two solutions with positive mass: $M^2 \approx 2.67 m^2$ and $M^2 \approx 4.66 m^2$.

4 Concluding remarks

As a first step towards construction of an effective field theory for adelic open scalar string, we have found a few Lagrangians which contain all corresponding n -adic Lagrangians ($n \in \mathbb{N}$). As a result one obtains that an infinite number of spacetime derivatives and related nonlocality are governed by the Riemann zeta function. Potentials are nonpolynomial. Tachyon mass spectra are determined by definite equations and they are contained in all the above cases. p -Adic Lagrangians can be easily restored from a zeta Lagrangian using just an inverse procedure for its construction.

This paper contains some basic classical properties of the introduced scalar field with zeta function nonlocality. There are rather many classical aspects which should be investigated. One of them is a systematic study of the equations of motion and nontrivial solutions. In the quantum sector it is desirable to investigate scattering amplitudes and make comparison with adelic string.

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